Ministry of Higher Education and Scientific Research
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Engineering Analysis

3^{rd} year

Method of solution of systems of Non-linear equations using Newton–Raphson method

suppose that we have the following system

$$f_1(x_1, x_2, x_3, \dots, x_k) = 0$$
$$f_2(x_1, x_2, x_3, \dots, x_k) = 0$$
$$f_3(x_1, x_2, x_3, \dots, x_k) = 0$$
$$\vdots$$
$$f_k(x_1, x_2, x_3, \dots, x_k) = 0$$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}$ and $\overrightarrow{f(x)} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_k \end{bmatrix}$

Then, the system of equation in vector form is

$$\overrightarrow{f(x)} = 0$$

since, Newton-Raphson method for single variable is

$$x_{n+1} = x_n - \frac{f(x_n)}{\bar{f}(x_n)}$$

where $\bar{f}(x_n)$ is the first derivative of $f(x_n)$

Hence the solution of the system can be written as

$$\overrightarrow{X_{n+1}} = \overrightarrow{X_n} - W^{-1}(x_n) \cdot \overrightarrow{f(x_n)}$$

where $W(x)$ is Jacobian matrix

$$W(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix}$$

Ex: Find the solution of the following system

$$x_1 + \frac{3}{\ln 10} \ln x_1 - x_2^2 = 0$$

$$2x_1^2 - x_1 x_2 - 5x_1 = -1$$

assume intial value for $x_1 = 3.4$ and $x_2 = 2.2$

<u>solution</u>

$$f_1(x_1, x_2) = x_1 + \frac{3}{\ln 10} \ln x_1 - x_2^2 = 0$$

$$f_2(x_1, x_2) = 2x_1^2 - x_1 x_2 - 5x_1 + 1 = 0$$

$$\vec{x_0} = \begin{bmatrix} 3.4 \\ 2.2 \end{bmatrix}, \quad \vec{f}(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \Rightarrow \vec{f}(x_0) = \begin{bmatrix} 0.1544 \\ -0.36 \end{bmatrix}$$

$$W(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{1.303}{x_1} & -2x_2 \\ 4x_1 - x_2 - 5 & -x_1 \end{bmatrix}$$

$$W(x_0) = \begin{bmatrix} 1.3832 & -4.4 \\ 6.4 & -3.4 \end{bmatrix}$$

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$$\overset{-1}{W}(x_0) = \frac{1}{23.4}\begin{bmatrix} -3.4 & 4.4 \\ -6.4 & 1.3832 \end{bmatrix}, \quad \vec{X}_{n+1} = \vec{X}_n - \overset{-1}{W}(x_n)\,\vec{f}(x_n)$$

if $\boxed{n=0}$

$$\vec{X}_1 = \vec{X}_0 - \overset{-1}{W}(x_0)\,\vec{f}(x_0)$$

$$\vec{X}_1 = \begin{bmatrix} 3.4 \\ 2.2 \end{bmatrix} - \frac{1}{23.4}\begin{bmatrix} -3.4 & 4.4 \\ -6.4 & 1.3832 \end{bmatrix}\begin{bmatrix} 0.1544 \\ -0.36 \end{bmatrix} = \begin{bmatrix} 3.4892 \\ 2.2633 \end{bmatrix}$$

$\boxed{n=1}$

$$\vec{X}_2 = \vec{X}_1 - \overset{-1}{W}(x_1) \cdot \vec{f}(x_1) = \begin{bmatrix} 3.4891 \\ 2.2621 \end{bmatrix}$$

$\boxed{n=2}$

$$\vec{X}_3 = \begin{bmatrix} 3.4875 \\ 2.2616 \end{bmatrix}$$

Ex: using the newton method, solve the following systems

$$x^2+y^2+z^2=1$$
$$2x^2+y^2-4z=0$$
$$3x^2-4y+z^2=0$$

Assume $x_0 = y_0 = z_0 = 0.5$

<u>solution</u>

$$\vec{X}_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$\vec{f}(x) = \begin{bmatrix} x^2+y^2+z^2-1 \\ 2x^2+y^2-4z \\ 3x^2-4y+z^2 \end{bmatrix}$$

$$W(X) = \begin{bmatrix} 2x & 2y & 2z \\ 4x & 2y & -4 \\ 6x & -4 & 2z \end{bmatrix}$$

$$\vec{f}(x_0) = \begin{bmatrix} -0.25 \\ -1.25 \\ -1 \end{bmatrix}$$

$$W(x_0) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 3 & -4 & 1 \end{bmatrix} \Rightarrow W^{-1}(x_0) = \frac{1}{40}\begin{bmatrix} 15 & 5 & 5 \\ 14 & 2 & -6 \\ 11 & -7 & 1 \end{bmatrix}$$

$$\vec{x}_{n+1} = \vec{x}_n - W^{-1}(x_n)\,\vec{f}(x_n)$$

$$\boxed{n = 0}$$

$$X_1 = \vec{X}_0 - W^{-1}(x_0)\,\vec{f}(x_0)$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 15 & 5 & 5 \\ 14 & 2 & -6 \\ 11 & -7 & 1 \end{bmatrix}\begin{bmatrix} -0.25 \\ -1.25 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.875 \\ 0.5 \\ 0.375 \end{bmatrix}$$

$$\therefore \vec{X}_1 = \begin{bmatrix} 0.875 \\ 0.5 \\ 0.375 \end{bmatrix} \Rightarrow \vec{f}(x_1) = \begin{bmatrix} 0.15625 \\ 0.2815 \\ 0.4375 \end{bmatrix}$$

$$W(x_1) = \begin{bmatrix} 1.75 & 1 & 0.75 \\ 3.5 & 1 & -4 \\ 5.2 & -4 & 0.75 \end{bmatrix}$$

$$W^{-1}(X_1) = \frac{1}{64.75}\begin{bmatrix} 15.25 & 3.75 & 4.75 \\ 23.625 & 2.625 & -5.62 \\ 19.25 & -12.25 & 1.75 \end{bmatrix}$$

$$\vec{X}_2 = \begin{bmatrix} 0.875 \\ 0.5 \\ 0.375 \end{bmatrix} - \frac{1}{6475}\begin{bmatrix} 15.25 & 3.75 & 4.75 \\ 23.625 & 2.625 & -5.62 \\ 19.25 & -12.25 & 1.75 \end{bmatrix}\begin{bmatrix} 0.15625 \\ 0.2815 \\ 0.4375 \end{bmatrix}$$

$$\vec{X}_2 = \begin{bmatrix} 0.7898 \\ 0.49662 \\ 0.36993 \end{bmatrix}$$

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$$\vec{X_3} = \vec{X_2} - W^{-1}(X_2) \cdot \vec{f}(X_2)$$

$$\vec{X_3} = \begin{bmatrix} 0.78521 \\ 0.49662 \\ 0.36992 \end{bmatrix} \Rightarrow \vec{f}(X_3) = \begin{bmatrix} 0.00001 \\ 0.00064 \\ 0.00005 \end{bmatrix}$$

$\therefore\ x \approx 0.78521\ ,\ y \approx 0.49664\ ,$ and $z \simeq 0.36992$

END OF LECTURE

ANY QUESTIONS?